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Letter to the Editors

Notes on the Article "Local Error of Difference Approximations to Hyperbolic Equations"*

In their paper "Local Error of Difference Approximation to Hyperbolic Equations" [1] Orszag and Jayne give the following local error behavior:

$$E_2(\lambda, t) = u(x, t) - u_2(x, t)$$
$$= D\,\mathrm{sgn}(\lambda) \int_{|\lambda/3^{1/3}|}^{\infty} Ai(t\,\mathrm{sgn}(\lambda))\, dt, \qquad (1)$$

where $Ai(x)$ is the Airy function and $\lambda = (6/h^2t)^{1/3}(x - t)$, for the semidiscretized approximation

$$\frac{\partial u_2}{\partial t} + \frac{u_2(x + h, t) - u_2(x - h, t)}{2h} = 0 \qquad (2)$$

to the wave equation when the solution exhibits a jump discontinuity of magnitude D at $x = t$. Note that Eq. (1) corrects Eq. (16) in Ref. [1] to read $|\lambda/3^{1/3}|$ instead of $|\lambda/3^{1/2}|$.

Orszag and Jayne obtain their result by solving the semi-discretized equation by Fourier analysis and then approximating the resultant sum by an integral.

It will be shown here that $E_2(\lambda, t)$ is essentially due to the contribution of the first truncation error term in a Taylor series expansion of the central difference approximation to the partial derivative, $\partial u/\partial x$. This term gives rise to the dispersive character of the semidiscretized solution, and thus to grid oscillations behind the main disturbance. It is these oscillations that must be filtered.

By expanding Eq. (2) and neglecting terms with orders higher than h^2, we obtain

$$\frac{\partial u_2}{\partial t} + \frac{\partial u_2}{\partial x} + \frac{1}{3!} h^2 \frac{\partial^3 u_2}{\partial x^3} = 0. \qquad (3)$$

The solution of (3) with a jump discontinuity of magnitude D at $x = t$ is obtained by the method of matched asymptotic expansions [2]. To do so, Eq. (3) will be transformed to the wave coordinate

$$\xi = x - t,$$

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giving

$$\frac{\partial u_2}{\partial t} = -\frac{1}{3!} h^2 \frac{\partial^3 u_2}{\partial \xi^3}.$$

The outer expansions with ξ fixed as $h \to 0$ are given by

$$u_2^0(\xi < 0, t; h) = \text{const} = \tilde{u} + D$$

and

$$u_2^0(\xi > 0, t; h) = \text{const} = \tilde{u}.$$

The inner expansion is the solution of

$$\frac{\partial u_2^i}{\partial t} = -\frac{1}{6} \frac{\partial^3 u_2^i}{\partial \eta^3},$$

with the stretched variable

$$\eta = \xi / h^{2/3}.$$

The appropriate solution is given by

$$u_2^i(\xi, t; h) = \alpha \int_0^{\eta(2/t)^{1/3}} Ai(\tau)\, d\tau + \beta,$$

where α and β are constants to be obtained by matching with the outer expansions. Following the matching principle of

$$\lim_{\xi \to 0} u_2^0(\xi, t; h) = \lim_{\eta \to \infty} u_2^i(\xi, t; h),$$

we obtain

$$u_2(\xi, t; h) = \tilde{u} + D \left\{ \frac{1}{3} - \int_0^{\xi(2/h^2 t)^{1/3}} Ai(\tau)\, dt \right\} + O(h^{2/3}).$$

Correspondingly,

$$E_2(\xi, t; h) \sim D \left\{ \tfrac{1}{3} - \int_0^{\xi(2/h^2 t)^{1/3}} Ai(\tau)\, d\tau \right\}.$$

We thus obtain the desired result, namely, that $E_2(\xi, t; h)$ is due to the contribution of the first truncation error term in a Taylor series expansion of the central difference approximation to $\partial u/\partial x$.

It is noteworthy that the application of the method of matched asymptotic expansions can greatly improve our understanding of error wave propagations.

References

1. S. A. Orszag and L. W. Jayne, *J. Comp. Phys.* **14** (1974), 93.
2. A. H. Nayfeh, "Perturbation Methods," Chapter 4, John Wiley and Sons, New York, 1973.

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